J Eng Math (2008) 61:133–141
DOI 10.1007/s10665-007-9202-2

Thermal stress-focusing effect in a spherical Zirconia inclusion with dynamically transforming strains

Toshiaki Hata · Naobumi Sumi

Received: 5 October 2006 / Accepted: 5 October 2007 / Published online: 27 October 2007

Abstract Some composite materials, such as Zirconia-toughened ceramics, are remarkable materials which have high strength, a high elastic modulus, and an improved toughness, etc. These good qualities are made possible through the stress-induced phase transformation of composite particles, which is accompanied by an impact cooling. When a spherical inclusion in an infinite elastic domain is suddenly subjected to an instantaneous phase transformation, stress waves occur at the surface of a spherical inclusion at the moment thermal impact is applied. The wave may accumulate at the center and show stress-focusing effects, even though the initial stress may be relatively small. This paper analyzes the thermal stress-focusing effect caused by the instantaneous anisotropic phase transformations in the spherical Zirconia inclusion. By use of ray theory, the numerical results give a clear indication of the mechanism of stress-focusing in an inclusion embedded in an infinite elastic medium.

Keywords Thermal stress-focusing effect · Phase transformation · Solid mechanics · Spherical inclusion · Wave propagation

1 Introduction

The transformation toughening of ceramics has attracted considerable attention in several works [1] towards the end of the twentieth century. The mechanism in the toughening of ceramics is the stress-induced phase transformation of a Zirconia particle [2], which is accompanied by volumetric expansion under cooling. Due to this expansion, the composite material consisting of Zirconia particles within a brittle matrix becomes more resistant to thermal fracture. However, the study of the interaction between a thermal shock and a possible phase transformation in the transformational-toughened ceramics with Zirconia particles has remained untouched so far.

In this paper a phenomenological model is proposed to describe the situation, which involves the interaction between a thermal shock and a dynamic inhomogeneity with a stress-induced martensitic transformation in a spherical particle of Zirconia embedded in an infinite elastic matrix.

When an infinite elastic medium with a spherical inclusion of Zirconia is suddenly subjected to impact cooling, stress waves occur at the surface of the spherical inclusion at the moment of thermal impact. The stress wave in an inclusion proceeds radially inward to the center of the inclusion. The wave may accumulate at the center and

T. Hata (✉) · N. Sumi
Faculty of Education, Shizuoka University, 836 Oya Suruga-ku, Shizuoka City 422-8529 Japan
e-mail: toshhata@silk.ocn.ne.jp

show the thermal stress-focusing effect, even though the initial thermal stress may be relatively small. Stress waves, which develop following rapid uniform cooling of linear elastic spheres, display a thermal stress-focusing effect as they proceed radially towards the center in this geometry. When the solid is cooled instantaneously through the phase-transformation point of Zirconia, the stress-focusing effect should occur in the spherical inclusion by the phase-transformational expansion and thermal contraction. It is important to note that the stress-focusing effect induced by the phase transformation may reduce the thermal stress-focusing effect in a spherical inclusion of Zirconia because of the antithetical volume change.

Regarding the study of the stress-focusing effect in a sphere, a closed-form solution by using the Laplace transformation is given in the form of a series expansion in the Laplace-transformed space. The inverse Laplace transformation of the solution, however, is very complicated. Recently Hata [3] obtained an exact solution to the problem of stress-focusing in a uniform heated solid sphere by applying ray theory.

Hata [4] solved, in an exact manner by using ray integrals, the effects of thermal stress-focusing in a spherical inclusion embedded in an infinite medium caused by an instantaneous phase transformation. The results give a clear indication of the mechanism of stress-focusing in an inclusion embedded in an infinite elastic medium. However, the foregoing analysis regarding the stress-focusing effect in an inclusion is limited to the isotropic phase transformation.

This paper analyzes the interaction between the thermal stress-focusing effect and the stress-focusing effect by the anisotropic phase transformation in a spherical inclusion of Zirconia embedded in an infinite medium subjected to thermal impact. By using ray theory, the Laplace-transformed solutions of stress waves in a spherical inclusion and in an infinite medium are separated into rays according to the ray paths of multiply reflected waves. The inverse transform of each ray leads to an exact solution of the transient response up to the arrival time of the next ray.

Following the ray method, we clarify the thermal stress-focusing effect in a spherical inclusion of Zirconia with the anisotropic phase transformation caused by the thermal impact. It should be noted that the successive stress-wave fronts occur in an infinite elastic medium corresponding with the stress-focusing effects in a spherical inclusion.

2 Formulation of the problem

The geometry of the problem is shown in Fig. 1. The medium and the inhomogeneity are denoted by M and I, respectively. Consider an infinite isotropic elastic medium of M containing a spherical inclusion of I with an eigenstrain (or transformation strain) $e^*_{ij}(\in \Omega)$. The coefficients of thermal expansion are α_I and α_M for the inclusion and the medium, respectively.

The governing equations for an inclusion [5] are

$$\sigma^I_{ij} = \rho_{0I}\ddot{u}^I_i, \quad \sigma^I_{ij} = C^I_{ijkl}(e^I_{kl} - e^*_{kl} - \delta_{kl}\alpha_I T), \quad e^I_{kl} = (u^I_{k,l} + u^I_{l,k})/2, \quad (i, j, k, l = 1, 2, 3) \tag{1}$$

where ρ_{0I}, α_I, δ_{ij} are the mass density, the coefficient of thermal expansion in an inclusion, the Kronecker delta, respectively, and C^I_{ijkl} is the elastic tensor of the inclusion as follows:

$$C^I_{ijkl} = \mu_I(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda_I\delta_{ij}\delta_{kl}, \tag{2}$$

where λ_I and μ_I are Lame's constants and the comma denotes differentiation by a following variable.

The following eigenstrain $e^*_{ij}(\mathbf{x}, t)$in the inclusion is considered as

$$e^*_{kl}(\mathbf{x}, t) = e^*_{kl}(\mathbf{x})f(t), \quad e^*_{kl}(\mathbf{x}) = \begin{cases} e^*_{kl} & \mathbf{x} \in \Omega \\ 0 & \mathbf{x} \in (I - \Omega) \end{cases}, \tag{3}$$

where $f(t)$ is a time function. The formulation of the problem in an infinite medium is

$$\sigma^M_{ij,j} = \rho_{0M}\ddot{u}^M_i, \quad \sigma^M_{ij} = C^M_{ijkl}(e^M_{kl} - \delta_{kl}\alpha_M T), \quad e^M_{kl} = (u^M_{k,l} + u^M_{l,k})/2. \tag{4}$$

For a medium with a spherical inclusion, the boundary conditions on the interface of $r = a$ are

$$\sigma^I_{ij}n_j = \sigma^M_{ij}n_j, \quad u^I_i = u^M_i \tag{5}$$

and the additional condition is that the displacement of the infinite medium at infinity is $u^M_i = 0$.

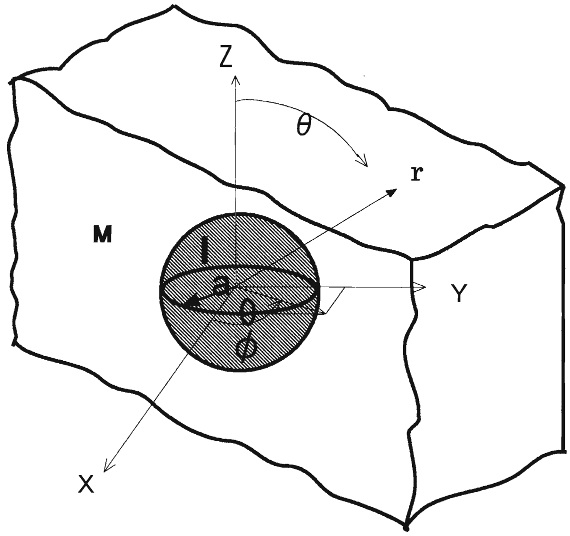

Fig. 1 Coordinate system of a spherical inclusion embedded in an infinite elastic medium

The medium with a spherical inclusion is at rest prior to time $t = 0$ and the initial conditions of displacement are

$$u_i^I(r,t) = u_i^I(r,t)_{,t} = 0, \quad u_i^M(r,t) = u_i^M(r,t)_{,t} = 0 \tag{6}$$

3 Formulation of a stress problem in an inclusion with anisotropic eigenstrains

An elastic medium with an elastic inclusion is at rest prior to time $t = 0$ and for $t > 0$ the medium is instantaneously cooled to the uniform temperature T_0. The boundary and initial conditions are given by Eqs. 5 and 6, respectively. The temperature distribution is assumed to have the following form

$$T(r,t) = -T_0 H(t) \quad \text{at } t = 0, \tag{7}$$

where $H(t)$ is the Heaviside step function.

The associated thermal stresses and strains of Eq. 7 in the spherical inclusion are

$$\sigma_{rI}^T = \sigma_{\vartheta I}^T = \sigma_{\varphi I}^T = (3\lambda_I + 2\mu_I)\alpha_I T_0 H(t), \quad e_{rI}^T = e_{\vartheta I}^T = e_{\varphi I}^T = -\alpha_I T_0 H(t). \tag{8}$$

The stress–strain relations without thermal strains in a spherically symmetric coordinate system take the form

$$\sigma_{rI} = 2\mu_I(e_r - e_r^*) + \lambda_I(e - e^*), \quad \sigma_{\vartheta I} = 2\mu_I(e_\theta - e_\theta^*) + \lambda_I(e - e^*), \quad \sigma_{\varphi I} = 2\mu_I(e_\varphi - e_\varphi^*) + \lambda_I(e - e^*), \tag{9}$$

where $\sigma_{\vartheta I} = \sigma_{\varphi I}$.

The strain-displacement relations of Eq. 1 are

$$e_r = \frac{\partial u_I}{\partial r}, \quad e_\theta = e_\varphi = \frac{u_I}{r}. \tag{10}$$

The equation of motion of Eq. 1 is given by

$$\frac{\partial \sigma_{rI}}{\partial r} + \frac{2(\sigma_{rI} - \sigma_{\vartheta I})}{r} = \rho_{0I} \frac{\partial^2 u_I}{\partial t^2}. \tag{11}$$

Therefore the equation of motion is obtained as

$$\frac{\partial^2 u_I}{\partial r^2} + \frac{2}{r}\frac{\partial u_I}{\partial r} - \frac{2u_I}{r^2} - \frac{1}{c_I^2}\frac{\partial^2 u_I}{\partial t^2} = \frac{1 - 2\nu_I}{1 - \nu_I}\left\{\frac{\partial e_r^*}{\partial r} + \frac{\nu_I}{1 - 2\nu_I}\frac{\partial e^*}{\partial r} + \frac{2\left(e_r^* - e_\theta^*\right)}{r}\right\}, \tag{12}$$

where c_I is the dilatational wave speed denoted as $c_I = \sqrt{(\lambda_I + 2\mu_I)/\rho_{0I}}$.

In the analysis the anisotropic eigenstrains of phase transformation in Eq. 3 are given by

$$e_r^* = \epsilon_1 r f(t) \quad e_\theta^* = e_\varphi^* = \epsilon_2 r f(t), \quad (0 < r \leq r_0),$$
$$e_r^* = e_\theta^* = e_\varphi^* = 0, \quad (r_0 < r \leq a), \tag{13}$$

where ϵ_1 and ϵ_2 are constants.

Under the action of phase transformation, it is assumed that the eigenstrains expand instantaneously along its radial direction self-similarly. Therefore, the function $f(t)$ is given by

$$f(t) = H(t), \tag{14}$$

where H(t) is the Heaviside step function. The associated radial eigendisplacement and the associated eigenstresses in a spherical inclusion are

$$u_I^p(r,t) = \frac{\epsilon_p r^2}{4} H(t), \tag{15}$$

$$\sigma_{rI}^p(r,t) = r\{-2\epsilon_2\lambda_I + \epsilon_p(\lambda_I + \mu_I) - \epsilon_1(\lambda_I + 2\mu_I)\}H(t), \tag{16}$$

where

$$\epsilon_p = \frac{(3-5\nu)\epsilon_1 + 2(3\nu_I - 1)\epsilon_2}{1-\nu_I}. \tag{17}$$

If $f(t) = H(t)$ and $r_0 = a$, the corresponding displacement and radial stress at the boundary of $r = a$ at $t = 0$ are

$$u_I^p \mid_{r=a} = \frac{\epsilon_p a^2}{4} H(t), \quad \sigma_{rI}^p \mid_{r=a} = a\{-2\epsilon_2\lambda_I + \epsilon_p(\lambda_I + \mu_I) - \epsilon_1(\lambda_I + 2\mu_I)\}H(t) \tag{18}$$

These displacements and stresses of Eqs. 8 and 18 do not satisfy the boundary conditions of Eq. 5 on the interface between the spherical inclusion and the infinite medium. In order to satisfy the boundary conditions, we have to solve an ordinary dynamic stress problem.

Here, when we introduce the displacement potential function Φ_I^S defined as $u_I^S = \frac{\partial \Phi_I^S}{\partial r}$, the function Φ_I^S must satisfy the equation

$$\frac{\partial^2 \Phi_I^S}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi_I^S}{\partial r} - \frac{1}{c_I^2}\frac{\partial^2 \Phi_I^S}{\partial t^2} = 0. \tag{19}$$

Applying the Laplace transform to Eq. 19, we can find

$$\bar{\Phi}_I^S(r,p) = C_1\left\{h_0^{(1)}(\mathrm{i}pr/c_I) + h_0^{(2)}(\mathrm{i}pr/c_I)\right\}, \tag{20}$$

where $h_0^{(1)}(z) = -\mathrm{i}e^{\mathrm{i}z}/z$ and $h_0^{(2)}(z) = \mathrm{i}e^{-\mathrm{i}z}/z$. We find the displacement and the associated radial stress in a spherical inclusion as

$$\bar{u}_I^S(\rho,p) = C_1\left\{h_0^{(1)}(\mathrm{i}p^*\rho)_{,\rho} + h_0^{(2)}(\mathrm{i}p^*\rho)_{,\rho}\right\}/a,$$
$$\bar{\sigma}_{rI}^S(\rho,p) = C_1\left\{-\frac{4\mu_I}{\rho}(\mathrm{i}p^*)h_0^{(1)\prime}(\mathrm{i}p^*\rho) + \rho_{0I}p^2 h_0^{(1)}(\mathrm{i}p^*\rho)\right\}$$
$$+C_1\left\{-\frac{4\mu_I}{\rho}(\mathrm{i}p^*)h_0^{(2)\prime}(\mathrm{i}p^*\rho) + \rho_{0I}p^2 h_0^{(2)}(\mathrm{i}p^*\rho)\right\}, \tag{21}$$

where $p^* = pa/c_I$ and $\rho = r/a$.

4 Formulation of a stress problem in an infinite medium

An elastic medium with an elastic inclusion is at rest prior to time $t = 0$ and, for $t > 0$, the medium is instantaneously cooled to the uniform temperature T_0. The corresponding thermal stresses and strains of Eq. 7 for an elastic infinite medium are

$$\sigma_{rM}^T = \sigma_{\vartheta M}^T = \sigma_{\varphi M}^T = (3\lambda_M + 2\mu_M)\alpha_M T_0 H(t), \quad e_{rM}^T = e_{\vartheta M}^T = e_{\varphi M}^T = -\alpha_I T_0 H(t). \tag{22}$$

These stresses and displacements do not satisfy the boundary conditions on the interface between the spherical inclusion and the infinite medium. In order to satisfy the boundary conditions, we have to solve an ordinary dynamic stress problem.

The elastic medium with an elastic inclusion is at rest prior to time $t = 0$ and the initial conditions of displacement are given by Eq. 6. For $t > 0$, the boundary conditions on the interface of $r = a$ are given by Eq. 5 and the additional condition is that the displacement at infinity should be $u_M = 0$. Upon introduction of a displacement potential ϕ_M^S defined as $u_M = \frac{\partial \phi_M^S}{\partial r}$, the equation of motion may be expressed as

$$\frac{\partial^2 \phi_M^S}{\partial r^2} + \frac{2}{r}\frac{\phi_M^S}{\partial r} = \frac{1}{c_M^2}\frac{\partial^2 \phi_M^S}{\partial t^2}, \tag{23}$$

where c_M is the dilatational wave speed denoted as $c_M = \sqrt{(\lambda_M + 2\mu_M)/\rho_{0M}}$. Applying the Laplace transform to Eq. 23 and solving the transformed equations, we may give the displacement potential for the infinite medium by

$$\bar{\phi}_M^S(r, p) = C_2 h_0^{(1)}\left(\mathrm{i}\frac{p}{c_M} r\right). \tag{24}$$

Differentiating $\bar{\phi}_M^S(r, p)$ in Eq. 24 by r, we obtain the displacement and the associated radial stress of the infinite medium as

$$\bar{u}_M(r, p) = C_2 h_0^{(1)}\left(\mathrm{i}\frac{p}{c_M} r\right)_{,r}, \quad \sigma_{rM}^{\bar{S}} = C_2\left\{-\frac{\mathrm{i}4\mu_M p^* l_1}{\rho} h_0^{(1)\prime}\left(\mathrm{i}p^* l_1 \rho\right) + \rho_{0M} p^2 h_0^{(1)}\left(\mathrm{i}p^* l_1 \rho\right)\right\}, \tag{25}$$

where $p^* = pa/c_I,\ \rho = r/a$, and $l_1 = c_I/c_M$. The unknown constants C_1 and C_2 may be determined from the boundary conditions of Eq. 5 with Eqs. 21 and 25 as follows;

$$C_1(c_{11}' + c_{12}') + \bar{\sigma}_{rI}^P\ |_{r=a} + \bar{\sigma}_{rI}^T = C_2 c_{13}' + \bar{\sigma}_{rM}^T, \quad C_1 c_{21} + \bar{u}_I^p\ |_{r=a} + \bar{e}_{rI}^T a = C_2 \mathrm{i}p^* l_1 h_0^{(1)\prime}(\mathrm{i}p^* l_1)/a + \bar{e}_{rM}^T a, \tag{26}$$

where

$$\begin{aligned} c_{11}' &= -\frac{4\mu_I p^*}{a} h_0^{(1)\prime}(p^* a) + \rho_{0I} p^2 h_0^{(1)}(p^* a), \quad c_{12}' = -\frac{4\mu_I p^*}{a} h_0^{(2)\prime}(p^* a) + \rho_0 p^2 h_0^{(2)}(p^* a), \\ c_{13}' &= -\frac{4\mu_M p^*}{a} h_0^{(1)\prime}(p^* l_1 a) + \rho_{0I} p^2 h_0^{(1)}(p^* l_1 a), \quad c_{21} = -p^*\left\{h_0^{(1)\prime}(p^* a) + h_0^{(2)\prime}(p^* a)\right\}. \end{aligned} \tag{27}$$

Then Eq. 26 yields

$$C_1 = \frac{g_2(p)}{c_{11} + c_{12}}, \quad C_2 = \frac{g_3(p)}{c_{11} + c_{12}}, \tag{28}$$

where

$$\begin{aligned} g_1(p) &= \frac{c_{13}'}{\mathrm{i}l_1 p^* h_0^{(1)\prime}(\mathrm{i}l_1 p^*)}, \quad g_2(p) = \bar{\sigma}_{rM}^T - \bar{\sigma}_{rI}^T - \bar{\sigma}_{rI}^p|_{r=a} + g_1(p)\left\{e_M^{\bar{T}} - e_I^{\bar{T}} + u_I^P\ |_{r=a}\right\}/p, \\ g_3(p) &= \frac{\left(c_{11}' + c_{12}'\right)\left\{\bar{e}_I^T - \bar{e}_M^T + \bar{u}_I^p\ |_{r=a}\right\} - c_{21}\left\{\bar{\sigma}_{rM}^T - \bar{\sigma}_{rI}^T - \bar{\sigma}_{rI}^p|_{r=a}\right\}}{\mathrm{i}l_1 p^* h_0^{(1)\prime}(\mathrm{i}l_1 p^*) p}, \\ c_{11} &= c_{11}' - g_1(p)(\mathrm{i}p^*) h_0^{(1)\prime}(\mathrm{i}p^*), \quad c_{12} = c_{12}' - g_1(p)(\mathrm{i}p^*) h_0^{(2)\prime}(\mathrm{i}p^*) \end{aligned} \tag{29}$$

5 Stress-focusing effect in a spherical inclusion described by ray integrals

In order to analyze the wave propagation in a spherical inclusion, we apply the ray theory to Eq. 20. Substituting Eq. 28 in Eq. 20, we obtain

$$\overline{\Phi_I^S} = \frac{g_2(p)}{c_{11} + c_{12}}\left\{h_0^{(1)}(\mathrm{i}p^* \rho) + h_0^{(2)}(\mathrm{i}p^* \rho)\right\}. \tag{30}$$

Here we introduce the reflection coefficient R, which is defined as $R = -c_{11}/c_{12}$. By introducing the formula $1/(1-R) = 1 + R + R^2 + \cdots \quad (|R| \leq 1)$ and rewriting Eq. 30, we obtain the displacement potential for a spherical inclusion by using the formulas of $h_0^{(1)}(z) = -\mathrm{i}e^{\mathrm{i}z}/z$ and $h_0^{(2)}(z) = \mathrm{i}e^{-\mathrm{i}z}/z$ as

$$\overline{\Phi_I^S}(\rho, p) = \sum_{j=0}^{\infty} \bar{\Psi}_j^S(\rho, p), \tag{31}$$

where

$$\begin{aligned}
\bar{\Psi}_0^S(\rho, p) &= -e^{p^*(\rho-1)} \frac{l_1^2 u^{IM}(\lambda_M + 2\mu_M)p^{*2} + (l_1 p^* + 1)\left(4\mu_M u^{IM} + \sigma^{IM}\right)}{f_1(p)\rho}, \\
\bar{\Psi}_1^S(\rho, p) &= e^{-p^*(\rho+1)} \frac{l_1^2 u^{IM}(\lambda_M + 2\mu_M)p^{*2} + (l_1 p^* + 1)(4\mu_M u^{IM} + \sigma^{IM})}{f_1(p)\rho}, \\
\bar{\Psi}_j^S(\rho, p) &= R(p)\bar{\Psi}_{j-2}^S(\rho, p), \quad (j = 2, 3, 4, \ldots).
\end{aligned} \tag{32}$$

The notations u^{IM}, σ^{IM}, and the function $f_1(p)$ in Eq. 32 are given by

$$\begin{aligned}
u^{IM} &= (\bar{e}_I^T - \bar{e}_M^T)a + u_I^P \,|_{r=a}, \quad \sigma^{IM} = \bar{\sigma}_{rI}^T + \bar{\sigma}_{rI}^P \,|_{\rho=1} - \bar{\sigma}_{rM}^T, \\
f_1(p) &= l_1(\lambda_I + l_1\lambda_M + 2\mu_I + 2l_1\mu_M)p^{*3} + \{\lambda_I + 2\mu_I - l_1(l_1\lambda_M + 4\mu_I + 2l_1\mu_M - 4\mu_M)\}p^{*2} \\
&\quad + 4(l_1 - 1)(\mu_I - \mu_M)p^* + 4(\mu_I - \mu_M).
\end{aligned} \tag{33}$$

An inspection of Eq. 32 shows that the function $\bar{\Psi}_0^S(\rho, p)$ has a singularity of $O(\rho^{-1})$ at $\rho = 0$. Hence, the corresponding displacement has a singularity of $O(\rho^{-2})$ at $\rho = 0$ and the corresponding stresses have a singularity of $O(\rho^{-3})$ at $\rho = 0$. An inspection of Eq. 32 suggests that the jth-order ray has the same order of the singularity as the function $\bar{\Psi}_0^S(\rho, p)$ at $\rho = 0$. Therefore the stress-focusing effects may be observed in each of the rays. Since the inverse Laplace transforms of Eq. 32 are easily obtained by using the inverse Laplace formulas and the high-order terms of $\Psi_j^S(\rho, t)$ are obtained through the convolution integral, the displacement potential for the spherical inclusion is obtained from Eq. 31 as

$$\Phi_I^S(r, t) = \sum_{j=0}^{\infty} \Psi_j^S(r, t). \tag{34}$$

Finally, the total displacement and stresses in the spherical inclusion are

$$\begin{aligned}
&u_I(\rho, t) = u_I^S(\rho, t) + u_I^T(\rho, t) + u_I^P(\rho, t), \quad \sigma_{rI}(\rho, t) = \sigma_{rI}^S(\rho, t) + \sigma_{rI}^T(\rho, t) + \sigma_{rI}^P(\rho, t), \\
&\sigma_{\theta I}(\rho, t) = \sigma_{\theta I}^S(\rho, t) + \sigma_{\theta I}^T(\rho, t) + \sigma_{\theta I}^P(\rho, t).
\end{aligned} \tag{35}$$

6 Wave propagation in an infinite medium obtained by ray integrals

In order to analyze wave propagation in an infinite medium, we apply the ray theory to Eq. 24. Substituting Eq. 28 in Eq. 24, we obtain

$$\overline{\phi_M^S}(\rho, p) = \frac{g_3(p)}{c_{11} + c_{12}} h_0^{(1)}(\mathrm{i}l_1 p^* \rho). \tag{36}$$

Here, introducing the reflection coefficient R which is defined as $R = -c_{11}/c_{12}$ and rewriting Eq. 36, we obtain the displacement potential for the infinite elastic medium as

$$\overline{\phi_M^S}(\rho, p) = \frac{g_3(p)}{c_{12}}[1 + R + R^2 + \cdots]h_0^{(1)}\left(\mathrm{i}l_1 p^* \rho\right) = \sum_{j=0}^{\infty} \bar{\varphi}_j^S(\rho, p), \tag{37}$$

where

$$
\begin{aligned}
\bar{\varphi}_0^S(\rho,p) &= -\mathrm{e}^{-l_1 p^*(\rho-1)} \frac{u^{IM}(\lambda_I+2\mu_I)p^{*2}+(p^*+1)(4\mu_I u^{IM}+\sigma^{IM})}{f_1(p)\rho}, \\
\bar{\varphi}_0^S(\rho,p) &= \mathrm{e}^{-l_1 p^*\rho+l_1-2} \frac{u^{IM}(\lambda_I+2\mu_I)p^{*2}+(p^*+1)(4\mu_I u^{IM}+\sigma^{IM})}{f_1(p)\rho}, \\
\bar{\varphi}_j^S(\rho,p) &= R(p)\bar{\varphi}_{j-2}^S(\rho,p), \quad (j=2,3,4,\ldots).
\end{aligned} \tag{38}
$$

Since the inverse Laplace transforms of Eq. 38 are easily obtained by using the inverse Laplace formulas, the high-order terms of $\varphi_j^S(\rho,t)$ are obtained through the convolution integral. Therefore the displacement potential for an infinite medium is derived from Eq. 37 as

$$
\phi_M^S(\rho,t) = \sum_{j=0}^{\infty} \varphi_j^S(\rho,t). \tag{39}
$$

Finally, the total displacement and stresses in the infinite medium are

$$
u_M(\rho,t) = u_M^S(\rho,t)+u_M^T(\rho,t), \quad \sigma_{rM}(\rho,t) = \sigma_{rM}^S(\rho,t)+\sigma_{rM}^T(\rho,t), \quad \sigma_{\theta M}(\rho,t) = \sigma_{\theta M}^S(\rho,t)+\sigma_{\theta M}^T(\rho,t). \tag{40}
$$

7 Numerical results and discussion

To show the mechanism in the toughening of ceramics subjected suddenly to an anisotropic phase transformation caused by impact cooling, we performed numerical calculations by using the material constants as given in [6] for a Z_rO_2 spherical inclusion embedded in an Al_2O_3 medium as

$$
\frac{E_M}{E_I} = 1.857, \quad \frac{c_I}{c_M} = 0.632, \quad \nu_I = 0.3, \quad \nu_M = 0.25, \quad \frac{\alpha_I}{\alpha_M} = 1.5, \quad \frac{u_I^P \mid r=a}{\alpha_I T_0} = 0.1. \tag{41}
$$

In order to show a numerical comparison of the phenomena, we make a calculation for two kinds of ratio of the tangential eigenstrain to the radial eigenstrain as follows:

Case I $= \epsilon_2/\epsilon_1 = 0.01$, Case II $= \epsilon_2/\epsilon_1 = 100$.

In case I the phase transformations in a sphere are affected mainly by the radial eigenstrains. In case II the phase transformations in a sphere are affected mainly by the tangential eigenstrains. The results of the numerical evaluation of the stress variation are illustrated in Figs. 2 to 5. In the figures we use the following nondimensional variables:

$$
\sigma_{\rho I} = \frac{\sigma_{rI}}{\rho_{0I} c_I^2 \epsilon_1}, \quad \tau = \frac{c_I t}{a} \tag{42}
$$

The behavior shape of the radial stress in a Zirconia inclusion with phase transformation as a function of time in Case I is illustrated in Fig. 2. The behavior shape of radial stress in a Zirconia inclusion with phase transformation as a function of time in Case II is illustrated in Fig. 3. In Fig. 2 we can observe that the waves reflected from the interface accumulate at the center of a spherical inclusion and give rise to very large stresses, even though the initial thermal stresses may be relatively small. The maximum of the radial stresses peaks out periodically at an interval of $\tau = 2$. In Fig. 3 these stress-focusing effects in a spherical inclusion are similar to those in Case I, whereas the peak stresses in an inclusion with phase transformation in Case II are much higher than those in Case I.

In order to investigate the stress-focusing effects at the center of a spherical inclusion, the radial stress variation at the center of a spherical inclusion as a function of time for Case I is shown in Fig. 4 and that for Case II is shown in Fig. 5.

In Fig. 4 the stress-focusing effects in Case I are observed at the times $\tau = 1, 3, 5, 7, 9$ periodically. As mentioned before in the discussion on Eq. 32, the radial stresses have a singularity of $O(\rho^{-3})$ at $\rho = 0$. In Fig. 5

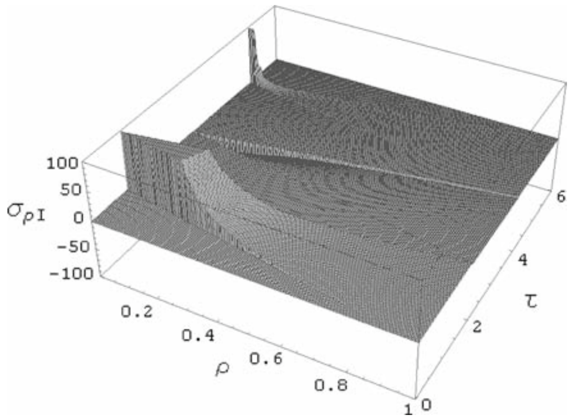

Fig. 2 The behavior shape of the radial stress $\sigma_{\rho I}$ in an inclusion with phase transformation in Case I

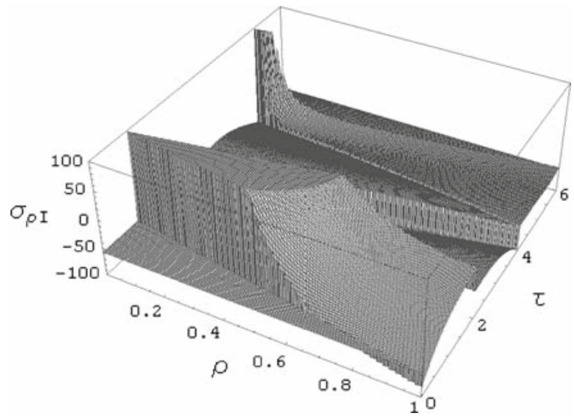

Fig. 3 The behavior shape of the radial stress $\sigma_{\rho I}$ in an inclusion with phase transformation in Case II

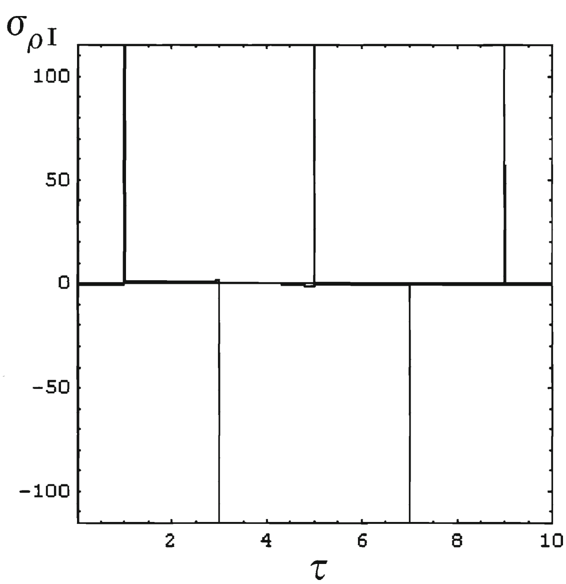

Fig. 4 Stress-focusing effects of the radial stress $\sigma_{\rho I}$ at the center in Case I

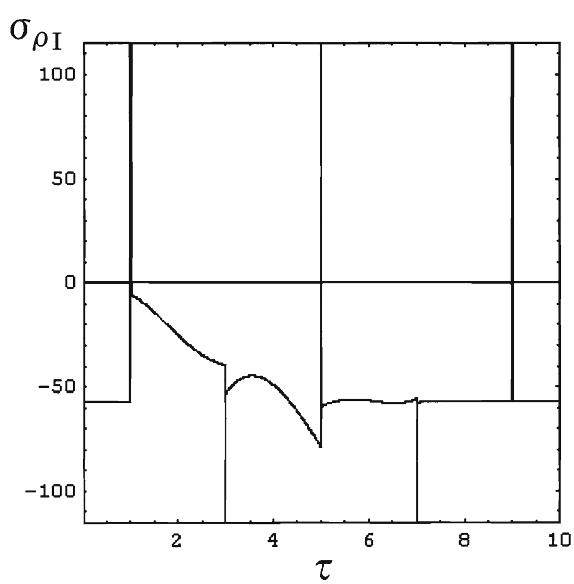

Fig. 5 Stress-focusing effects of the radial stress $\sigma_{\rho I}$ at the center in Case II

we also observe the stress-focusing effects at the times $\tau = 1, 3, 5, 7, 9$ periodically in Case II. The difference between Figs. 4 and 5 is the initial stress distribution. Therefore, the initial stress distribution caused by the phase transformation does not affect the stress-focusing effects at the center of the sphere. Comparing these results with the stress-focusing effects in a spherical inclusion with the isotropic phase transformation in [4], the same type of stress-focusing effects are observed.

8 Conclusions

The mechanism in the toughening of ceramics is the stress-induced phase transformation of a Zirconia particle, which is accompanied by volumetric expansion under the cooling process. Due to this expansion, the composite material consisting of Zirconia particles within a brittle matrix becomes more resistant to thermal fracture. However, the study of the interaction between a thermal shock and a possible phase transformation in the phase-transformational-toughened ceramics with Zirconia particles has remained untouched. In this paper a phenomenological model has been proposed to describe the situation, which involves the interaction between a thermal shock and a dynamic inhomogeneity with a stress-induced martensitic transformation in a spherical particle of Zirconia embedded in an infinite elastic matrix. When an infinite elastic medium with a spherical inclusion of Zirconia is suddenly subjected to an impact cooling, stress waves occur at the surface of spherical inclusion at the moment thermal impact is applied. The stress wave in an inclusion proceeds radially inwards to the center of the inclusion. The wave may accumulate at the center and show the thermal stress-focusing effects, even though the initial thermal stress may be relatively small.

In the paper, we conclude that the high-stress pulses induced by the stress-focusing effects in a spherical inclusion may cause cracks at the center of a spherical inclusion. Since these cracks at the center of a spherical inclusion may break an inclusion, the fracture may occur in an inclusion under dynamic inhomogeneity. In the steady state, however, the fracture in the matrix with a spherical inclusion may occur at the interface of an inclusion because of the volumetric expansion of the Zirconia inclusion. Therefore, we conclude that the stress-induced mechanism of phase transformation in the toughening of the Aluminum Oxide ceramics with a Zirconia inclusion in the steady state does not hold in the dynamic state.

References

1. Mikata Y, Nemat-Nasser S (1990) Elastic field due to a dynamically transforming spherical inclusion. J Appl Mech 57:845–849
2. Garvie C, Hannink RH, Pascoe RT (1975) Ceramic steel. Nature 258:703–704
3. Hata T (1991) Stress-focusing effect in a uniformly heated solid sphere. J Appl Mech 58:58–63
4. Hata T (2003) Stress-focusing effects in a spherical inclusion embedded in an infinite medium caused by instantaneous phase transformation. In: Watanabe K, Ziegler F (eds) IUTAM Symposium on dynamics and advanced materials and smart structures, Kluwer Publishers, pp 95–104
5. Mura T (1987) Micromechanics of defects in solids, 2nd edn. Martinus Nijhoff Publishers, Dordrecht
6. Lange FF (1982) Transformation toughening Part 4. Fabrication, fracture toughness and strength of $Al_2O_3 - Z_rO_2$. J Mater Sci 17:247–254